3-1: Isometric Transformations

Isometric Transformation: An isometric transformation is a transformation that preserves the distances and angles between a pre-image and its image. Put simply, in an isometric transformation the image is exactly the same size and shape as its pre-image.

Here are some examples of isometry...

Example #1
Translation (slide)

Example #2
Reflection (flip or mirror image)

Example #3
Rotation (turn)

Isometric transformations are more commonly referred to as "rigid transformations." That is because the shape of the pre-image does not change when transformed to its image, just like a steel pipe is referred to as rigid because when you move it around it keeps its shape. This is opposed to something like maple syrup, which does not hold its shape very well on its own.

In an isometric or "rigid" transformation, segments are taken to segments of the same measure, and angles are taken to angles of the same measure. That is to say that all the segments in the pre-image are the same length as their corresponding segments in the image. The same goes for the measures of all the pairs of corresponding angles. For example in #3 above, |xy|=|x'y'|, |yz|=|y'z'|, |zx|=|z'x'|, |cx|=|c'x'|, |cy|=|c'y'|, and |cz|=|c'z'|. Also, parallel lines are taken to parallel lines and perpendicular lines are taken to perpendicular lines. We will explore this more in another section.

Non-Isometric Transformation: A non-isometric transformation is a transformation that does not preserve the distances and angles between a pre-image and its image. So, in a non-isometric transformation the image is a different size or shape than its pre-image.

Here are some examples of non-isometry...

Example #1
Shear (pushed over)

Example #2
Stretch (Pulled)

Example #3
Dilation (changed size)

There are a lot more examples of non-isometric transformations but the above are the more recognizable examples.
Isometric Transformations: Translations

**Translation:** A translation is a transformation consisting of a constant offset with no rotation or distortion.

In other words, a translation is a transformation in which a geometric figure is "moved" so that it is not turned or changed in any way. Look at the example below...

Here is some of the language of transformations. Complete each sentence below.

1. A is taken to $A'$.
2. $CD$ is taken to $CD'$.
3. B maps to $B'$.
4. $\angle BCD$ maps to $\angle B'C'D'$.
5. $C'$ is the image of C.
6. Figure $A'B'C'D'$ is the image of Figure $ABCD$.
7. A translation results in an isometric transformation. Therefore, the image figure is exactly the same size and shape as the pre-image. Does the image of $ABCD$ appear to be the same size and shape as its pre-image? **Yes**.
A translation can be expressed by a function. Look at the triangle below. It has been translated according to the following function: \((x, y) \rightarrow (x+9, y-4)\). That is to say, that each point of the triangle has been translated 9 in the x direction (to the right), and -4 in the y direction (down).

Point A is at (-9,5) so using our function \((-9,5) \rightarrow (-9+9, 5-4)\) means that \(A'\) is at (0,1).

Point B is at (-6,8) so using our function \((-6,8) \rightarrow (-6+9, 8-4)\) means that \(B'\) is at (3,4).

Point C is at (-4,1) so using our function \((-4,1) \rightarrow (-4+9, 1-4)\) means that \(C'\) is at (3,4).

All the points that make up the line segments in the figures are also translated according to the rule.

1. Write a function that will map \(\triangle DEF\) to \(\triangle D'E'F'\):
\[ (x,y) \rightarrow (x-9, y+4) \]

How is this function similar to the one above?

They are both translations. Both isometric.
**Isometric Transformations: Reflections**

**Reflections:** A transformation in which every point from a figure maps to its mirror image on the other side of a line of reflection.

The line of reflection also becomes an axis of symmetry.

In the example below, ABCD was reflected through the y axis. We can use the notation: \( R_y \text{ axis} \).

![Diagram showing reflection](image)

1. In the reflection above, compare |AB| and its image |A'B'| by finding the lengths of each.

   \[
   AB = 4 \text{ units} \quad A'B' = 4 \text{ units}
   \]

2. Compare the lengths of the other segments in ABCD to their images in A'B'C'D'. You might need to use the Pythagorean theorem.

   \[
   AD = 6 \text{ units} \quad CB = \sqrt{20} \quad CD = \sqrt{40}
   \]

   \[
   A'D' = 6 \text{ units} \quad C'B' = \sqrt{20} \quad C'D' = \sqrt{40}
   \]

3. Is the reflection above an isometric transformation? In other words, are ABCD and A'B'C'D' exactly the same size and shape? Why?

   *Yes, because all the side lengths are equal.*
Directions: Refer to some of the problems on the previous page to help you make conjectures about the functions of rotations about the origin.

7. Reflect ABC through the x axis.
   a. What are the coordinates of the vertices of the original figure?
      A(2, 2) B(5, 4) C(4, 1)
   b. What are the coordinates of the vertices of A'B'C'?
      A'(2, -2) B'(5, -4) C'(4, -1)
   c. Explain in writing how the coordinates of ABC have been changed to create A'B'C' in this reflection through the x axis.
      *The y-value becomes negative*

8. Reflect QRS through the y axis.
   a. What are the coordinates of the vertices of the original figure?
      Q(1, 3) R(5, 3) S(1, -4)
   b. What are the coordinates of the vertices of Q'R'S'?
      Q'(1, -3) R'(-5, 3) S'(-1, -4)
   c. Explain in writing how the coordinates of QRS have been changed to create Q'R'S' in this reflection through the y-axis.
      *The x-value changes sign*

   d. Write a function that describes a reflection through the x axis.
      \[(x, y) \rightarrow (x, -y)\]

   d. Write a function that describes a reflection through the y axis.
      \[(x, y) \rightarrow (-x, y)\]
**Isometric Transformations: Rotations**

**Rotation**: A transformation that turns the plane through a given angle about (around) a given point.

In other words, a translation is a turn around a center point. The angle is called the **angle of rotation** and the point around which the plane is turned is called the **center point of rotation**.

But before we start spinning around, let's talk about angles and clocks.

<table>
<thead>
<tr>
<th>1. What kind of angle is this?</th>
<th>2. What kind of angle is this?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Right</strong></td>
<td><strong>Straight</strong></td>
</tr>
<tr>
<td>How many degrees does it measure?</td>
<td>How many degrees does it measure?</td>
</tr>
<tr>
<td>90°</td>
<td>180°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3. How many degrees are there in a quarter rotation clockwise?</th>
<th>4. How many degrees are there in a quarter rotation counter-clockwise?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="Image" alt="Quarter Rotation" /> 90°</td>
<td><img src="Image" alt="Quarter Rotation" /> 90°</td>
</tr>
<tr>
<td>Did this rotate the same direction as a clock or against it?</td>
<td>Did this rotate the same direction as a clock or against it?</td>
</tr>
<tr>
<td><strong>Same</strong></td>
<td><strong>Against</strong></td>
</tr>
<tr>
<td>So a rotation counter-clockwise means rotate <strong>the same direction</strong> as a clock.</td>
<td>So a rotation counter-clockwise means rotate <strong>against</strong> a clock.</td>
</tr>
<tr>
<td>We can also say this is a -90° rotation.</td>
<td>We can also say this is a 90° rotation.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5. How many degrees are there in a half rotation counter-clockwise?</th>
<th>6. How many degrees are there in a half rotation clockwise?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="Image" alt="Half Rotation" /> 180°</td>
<td><img src="Image" alt="Half Rotation" /> 180°</td>
</tr>
<tr>
<td>Did this rotate the same direction as a clock or against it?</td>
<td>Did this rotate the same direction as a clock or against it?</td>
</tr>
<tr>
<td><strong>Against</strong></td>
<td><strong>Same</strong></td>
</tr>
<tr>
<td>So a rotation counter-clockwise means rotate <strong>against</strong> a clock.</td>
<td>So a rotation counter-clockwise means rotate <strong>same direction</strong> a clock.</td>
</tr>
<tr>
<td>We can also say this is a 180° rotation.</td>
<td>We can also say this is a -180° rotation.</td>
</tr>
</tbody>
</table>
### Directions: Refer to some of the problems on the previous page to help you make conjectures about the functions of rotations about the origin.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Rotation Type</th>
<th>Original Figure</th>
<th>Rotated Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.</td>
<td>180° clockwise</td>
<td>B (-3, 4) U (-1, 1) G (-5, 2)</td>
<td>B' (3, -4) U' (1, -1) G' (5, -2)</td>
<td>Changing the signs of x &amp; y values</td>
</tr>
<tr>
<td>8.</td>
<td>90° clockwise</td>
<td>T (5, -3) O (-1, -4) E (2, 5)</td>
<td>T' (3, -5) O' (1, -4) E' (5, -2)</td>
<td>Switching the x &amp; y values &amp; x becomes negative</td>
</tr>
<tr>
<td>9.</td>
<td>90° counter-clockwise</td>
<td>D (6, 7) A (8, -1) Y (-3, 1)</td>
<td>D' (-7, 6) A' (-1, 8) Y' (-1, -3)</td>
<td>The x &amp; y values switch but the y becomes negative</td>
</tr>
<tr>
<td>10.</td>
<td>90° clockwise</td>
<td>(x, y) ⇒ (-y, x) ⇒ (-x, -y)</td>
<td>(x, y) ⇒ (y, -x) ⇒ (-x, -y)</td>
<td>Same coordinates!</td>
</tr>
</tbody>
</table>

Bill says if you rotate a figure 180 clockwise or counterclockwise you will get the same image. Sally says you won't. Who is correct? Why?

Bill, counterclockwise: (90°+90°) (x, y) ⇒ (-y, x) ⇒ (-x, -y)

Clockwise: (-90°+90°) (x, y) ⇒ (y, -x) ⇒ (-x, -y)

Same coordinates!
Counter-clockwise is the norm